ONE METHOD OF EVALUATING APPROXIMATE SOLUTIONS OF THE EQUATION OF NON-STATIONARY FILTRATION OF LIQUIDS AND GASES

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PMM Vol.25, No.4, 1961, pp. 756-760

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(Received April 10, 1961)

The exact solution of the nonlinear equation of the unsteady filtration of a liquid with a free surface in a layer with variable penetrability in a vertical direction involves great mathematical difficulties [1,2].

In the present article we use a theorem of comparison [3] to obtain simple evaluations of approximate solutions of an equation of one-dimensional filtration in a layer of variable penetrability.

1. Let us examine the known problem of the outflow of a liquid in plane waves into a layer with a zero level of ground waters [4].

Let us assume that the coefficient of filtration is a certain given bounded function of the vertical coordinate $k = k(z)$.

The differential equation of filtration for this case has the form

$$\frac{\partial}{\partial x}\left(\varphi(H)\frac{\partial H}{\partial x}\right) = m(H)\frac{\partial H}{\partial t} \qquad \left(\varphi(H) = \int_0^H k(z)\,dz\right) \qquad (1.1)$$

where $m(H)$ is the coefficient, variable in a vertical direction, of the saturation defect. $\phi(H)$ is clearly a finite non-negative function.

Let us examine similarity solutions of the form

$$H = H(u) \qquad (u = xt^{-1/2}) \qquad (1.2)$$

which correspond to the case of a momentary rise in the reservoir from H_2 to $H_1 = 1$: in the present instance it is obvious that $H_2 = 0$.

Substitution of (1.2) into (1.1) yields

$$\frac{d}{du}\left(\varphi(H)\frac{dH}{du}\right) = -\frac{u}{2} m(H)\frac{dH}{du} \tag{1.3}$$

With a coefficient of filtration which is independent of thickness $k = k_1$, Equation (1.1) degenerates into the well-known Boussinesq differential equation of filtration for the analogous boundary condition in the zero section.

Equation (1.3) permits an exact solution for certain special forms of function $\phi(H)$.

To obtain these solutions we will proceed in the following manner. Having integrated Equation (1.3) from zero to H, we obtain

$$\varphi(H)\frac{dH}{du}\Big|_H - \varphi(H)\frac{dH}{du}\Big|_{H=0} = -\frac{1}{2}\int_0^H um(H)\,dH$$

$$\varphi(H) = -\frac{1}{2}\frac{du}{dH} Q(H) \qquad \left(Q(H) = \int_0^H um(H)\,dH\right) \tag{1.4}$$

We took account of the fact that $\phi(H)\ dH/du = 0$ when $H = 0$, as a result of the continuity of distribution of the flow of ground waters and the tendency of this flow toward zero, when $x \to \infty (u \to \infty)$.

At this point, if we take a relationship of the form $u = u(H)$, we may determine the function $\phi(H)$ and, consequently, also the law of penetrability as a function of the thickness of the porous layer.

The above inverse method was used to find several exact solutions, on the basis of which we may evaluate approximate solutions.

An analysis of the exact solutions indicates that only those functions of the saturation distribution which are identically equal to zero when u is quite large correspond to bounded non-negative values of $\phi(H)$.

This latter feature is connected with the finite extent of the perturbation, proved in [5] for the case of the generalized Boussinesq-Leibenson equation, which gives a completely general statement of the problem. The finiteness of the velocity of propagation of the disturbance for similarity problems examined here will also emerge clearly from the following simple considerations.

In Equation (1.4) we will let H tend toward zero; then $\phi(H) \to 0$, $Q(H) \to 0$ (since the integral converges), and the fraction

$$\frac{du}{dH} = -\frac{2\varphi(H)}{Q(H)} \tag{1.5}$$

will become undetermined.

Investigation of this undetermined value yields

$$u'(0) = -\frac{2\varphi'(0)}{u(0)\,m(0)} = -\frac{2k(0)}{u(0)\,m(0)} \tag{1.6}$$

Let us assume that $k(0) \neq 0$, $m(0) \neq 0$. If the axis u were an asymptotic integral curve, we would have to have $u(0) = \infty$, $u'(0) = -\infty$; however, this would contradict (1.6).

2. We will prove that through point $u = 0$, $H = 1$ there may not pass more than one curve corresponding to conditions

$$H > 0,\ H' < 0 \text{ and } \lim \varphi(H)\,H' = 0 \quad \text{for } u \to \infty$$

Let us designate the coordinate of the saturation front by u^*. If another integral curve (other than $H(u)$) passes through the point (0.1), such that $H_1(u) \geqslant H(u)$, ($u_1(H) \geqslant u(H)$), there must exist a segment $[0, u_0]$ for which

$$-H_1'(u) \leqslant -H'(u), \qquad Q_1(H) > Q(H) \qquad \left(Q_1(H) = \int_0^H u_1 m(H)\,dH\right) \tag{2.1}$$

$$-\frac{1}{2}\frac{Q_1(H)}{H_1'(u)} > -\frac{1}{2}\frac{Q(H)}{H'(u)} = \varphi(H) \tag{2.2}$$

From (2.2) it follows that the curve $H_1(u)$ on the segment $[0, u_0]$ does not satisfy the integro-differential equation (1.4) and consequently does not satisfy the differential equation.

Let us suppose that the integral curve $H_1(u)$ intersects the curve $H(u)$ at several points $u_1, u_2, \ldots, u_n$. If, for the segment $[u_n u^*]$ the inequality $H_1(u) \geqslant H(u)$ is correct, then in $[u_n \ u_0]$, contained in the segment, the following must apply

$$-H_1'(u) \leqslant -H'(u),\ Q_1(H) > Q(H),\ -\frac{1}{2}\frac{Q_1(H)}{H_1'(u)} < -\frac{1}{2}\frac{Q(H)}{H'(u)} = \varphi(H) \tag{2.3}$$

and we again come to a contradiction. Thus, the curve $H_1(u)$ on the segment $[u_n u_0]$ does not satisfy the integro-differential equation (1.4) (or the differential equation (1.3)).

In an entirely similar way we can prove that there are no integral curves coming from the point (0.1) which would lie below the curve $H(u)$ or intersect it so that for points $u > u_n$ one obtains $H_1(u) < H(u)$. With the cases we have examined, we have exhausted all possible curves passing through the initial point (0.1) of the curve $H(u)$: as we can see, not one of these curves can be an integral curve of the differential equation (1.3).

3. Equation (1.4) may be reduced to the form

$$Q'' = -\frac{2m(H)\varphi(H)}{Q} + \frac{m'(H)}{m(H)}Q' \qquad (3.1)$$

Differentiation with respect to H is designated by a prime. We note that the flux of the filtration current in the section we are examining is

$$v = Q/2t^{1/2} \qquad (3.2)$$

Let the differential equation

$$Q'' = -\frac{2m_1(H)\varphi_1(H)}{Q} + \frac{m_1'(H)}{m_1(H)}Q' \qquad (3.3)$$

be given.

Let us further assume that $Q(H)$ and $Q_1(H)$ are the solutions to Equations (3.1) and (3.3), respectively. We will assume that

$$\varphi_1(H) \leqslant \varphi(H), \qquad u_1(1) = u(1) = 0, \qquad m_1(H) = m(H) \qquad (3.4)$$

The subsequent development is based on a theorem of comparison given in [3]. First, however, it is necessary to carry out the following arguments.

If $u_1(H) \geqslant u(H)$ on the segment $[0, H_0]$ of the H-axis, then in $[H_0, H_m]$, contained on that segment, $-u_1'(H) \geqslant -u^8(H)$, $Q_1(H) > Q(H)$ and $\phi_1(H) \geqslant \phi(h)$, which contradicts condition (3.4).

Therefore, the only such relative position of integral curves which is possible is $u(H)$ and $u_1(H)$, for which $u(0) \geqslant u_1(0)$. In the plane QH, $Q'(0) \geqslant Q_1(0)$ obviously corresponds to this latter condition. Thus, in every case in the neighborhood of $H = 0$, the difference $Q(H) - Q_1$ is positive and satisfies the relationship

$$Q(H) - Q_1(H) = [Q'(0) - Q_1'(0)]\,H + o(H) \qquad (3.5)$$

Let us further assume that the integral curves intersect at one point. Under these conditions, when $H = 1$, $Q_1(1) > Q(1)$, and, since $Q'(1) - Q_1'(1) = 0$ according to (3.4), then either at this point or in the interval between the point of intersection and $H = 1$, the difference $Q(H) - Q_1(H)$ will have to be a minimum, i.e.

$$Q''(H) - Q_1''(H) \geqslant 0 \qquad (3.6)$$

Having taken this into account, we obtain from (3.1) and (3.3)

$$Q''(H) - Q_1''(H) = -2\left(\frac{\varphi(H)}{Q(H)} - \frac{\varphi_1(H)}{Q_1(H)}\right) m(H) < 0 \qquad (3.7)$$

which contradicts the requirement (3.6). Furthermore, if there is a minimum at a point in the interval, then between $H = 1$ and that point $Q''(H) - Q_1''(H) = 0$, which is also impossible.

It is easy to see that, with an odd number of intersections of curves $Q(H)$ and $Q_1(H)$, $Q_1(1) > Q(1)$ and the above considerations remain valid.

With an even number of intersections in the interval $[0, 1]$, it is clear that there will be at least one point $H = H_m$, at which $Q(H) - Q_1(H)$ attains a minimum when $Q(H) < Q_1(H)$. When we use Equations (3.1) and (3.3) for this point, we can easily come to a contradiction. Thus, if we observe conditions (3.4) for the solutions of differential equations (3.1) and (3.3), we obtain the inequality

$$Q(H) \geqslant Q_1(H) \qquad (3.8)$$

Now let the two solutions $Q_1(H)$ and $Q_2(H)$ of the equations

$$Q'' = -\frac{2m_i(H)\varphi_i(H)}{Q} + \frac{m_i'(H)}{m_i(H)} Q' \qquad (i=1, 2) \qquad (3.9)$$

be known, where

$$\varphi_1(H) \leqslant \varphi_2(H), \ldots, \qquad u_1(1) = u_2(1) = 0, \ldots, \qquad m_1(H) = m_2(H) = m(H) \qquad (3.10)$$

Let us further assume fulfilment of the following condition:

$$\varphi_1(H) \leqslant \varphi(H) \leqslant \varphi_2(H) \qquad (3.11)$$

It is obvious that

$$Q_2(H) \geqslant Q(H) \geqslant Q_1(H) \qquad (3.12)$$

With a proper selection of distribution curves $u_1(H)$ and $u_2(H)$ (or of curves $Q_1(H)$ and $Q_2(H)$), it is possible in a number of cases to satisfy condition (3.11) by the inverse method given in Section 1 and, at the same time, to give a lower and upper estimate of the approximate solution of the differential equation (3.1), where $\phi(H)$ is replaced by its majorant or by its minorant.

4. Let us assume in Equation (1.1) that $m(H) = m_0 =$ const, and illustrate the evaluation method for that particular equation. The solution of examples on the basis of the more general equation (1.1) contributes nothing essential to the understanding of the method and at the same time leads to several complications.

Introducing a new variable $u = m_0^{1/2} x t^{-1/2}$ (for which we retain the previous designation), we obtain from (1.7), for the case $m(H) = m_0$

$$\varphi(H) = -\frac{1}{2} u' \int_0^H u\, dH \tag{4.1}$$

Let us specify the solution for the saturation distribution in the form

$$u = c_1 (1 - H) + c_2 (1 - H^2) \tag{4.2}$$

and determine $\phi(H)$ from (4.1) by the inverse method. By computations which we will omit here, we obtain for $\phi(H)$, according to Formula (4.1), the following value:

$$\varphi(H) = \left[\frac{1}{2} c_1 (c_1 + c_2)\right] H + \left[c_2 (c_1 + c_2) - \frac{1}{4} c_1^2\right] H^2 - \left[\frac{2}{3} c_1 c_2\right] H^3 - \left[\frac{1}{3} c_2^2\right] H^4 \tag{4.3}$$

By differentiating with respect to H, we obtain from (4.3) the following expression for penetrability:

$$k(H) = \left[\frac{1}{2} c_1 (c_1 + c_2)\right] + \left[2c_2 (c_1 + c_2) - \frac{1}{2} c_1^2\right] H - [2c_1c_2]\, H^2 - \left[\frac{4}{3} c_2^2\right] H^3 \tag{4.4}$$

For the simplest cases of $c_1 = 0$ and $c_2 = 0$, we obtain from (4.3) and (4.4)

$$\varphi(H) = c_2^2 \left(H^2 - \frac{1}{3} H^4\right), \qquad k(H) = 2c_2^2\left(H - \frac{2}{3} H^3\right) \tag{4.5}$$

$$\varphi(H) = \frac{1}{2} c_1^2 \left(H - \frac{1}{2} H^2\right), \qquad k(H) = \frac{1}{2} c_1^2 (1 - H) \tag{4.6}$$

Let us note that in the case of (4.5) the first derivative of the saturation at a front is equal to infinity; however, the flux at that point vanishes.

Examples. Let us compare the two solutions

$$\varphi_1(H) = H - \frac{1}{2} H^2, \qquad \varphi_2(H) = H \tag{4.7}$$

$\phi_1(H)$ corresponds to a linear change of penetrability with thickness from $k = 1$ at the base to $k = 0$ at the top, $\phi_2(H)$ to a constant value of penetrability $k = 1$.

The solution for $\phi_1(H)$ according to (4.6) has the form

$$H_1 = 1 - \sqrt{\frac{m_0}{2t}}\, x \tag{4.8}$$

For $\phi_2(H)$ we obtain the solution from [4]

$$H_2 = -c\left(\frac{u}{\sqrt{2}} - c\right) - \frac{1}{4}\left(\frac{u}{\sqrt{2}} - c\right)^2 - \frac{1}{72c}\left(\frac{u}{\sqrt{2}} - c\right)^3 + \dots \quad (c = 1.14277\dots) \qquad (4.9)$$

Substituting the values H_1 and H_2 from (4.8) and (4.9) into (3.3), we obtain, when $u = 0$,

$$v_1 = 0.5\sqrt{\frac{m_0}{2t}}, \qquad v_2 = 0.628\sqrt{\frac{m_0}{2t}}, \qquad \frac{v_2}{v_1} = 1.256 \qquad (4.10)$$

It is evident from (4.10) that the difference in the rates of the filtration flow is 26%, although the penetrability curves strongly differ from each other.

Closer values are obtained when

$$\varphi_2(H) = H - \frac{1}{2}H^2, \qquad \varphi_1(H) = \frac{1}{2}H \qquad (4.11)$$

Omitting the intermediate calculations, we obtain

$$v_2/v_1 = 1.13$$

If $Q = 2m_0^{-1/2}t^{1/2}v$ is the solution of the differential equation (3.2) (with m = const) and condition (3.12) is fulfilled, then, for the first and second examples

$$1 < v/v_1 < 1.256, \qquad 1 < v/v_1 < 1.13$$

respectively.

It follows that if, in the differential equation (3.2), $\phi(H)$ is replaced by its majorant or minorant, the error permitted in the calculation of v will not exceed 26% (in the first example) and 13% (in the second).

Let us select $\phi_2(H)$ in the form of a polynomial (4.4), whereby we assume

$$2c_2(c_1 + c_2) - \frac{1}{4}c_1^2 = 0, \qquad \frac{1}{2}c_1(c_1 + c_2) = 1$$

The calculations for $\phi_2(H)$ and v_2 yield the following values:

$$\varphi_2(H) = H - 0.230H^3 - 0.024H^4, \qquad v_2 = 0.411\sqrt{m_0/t}$$

For $\phi_1(H) = H \geqslant \phi_2(H)$, we have, according to [4]

$$v_1 = 0.628\sqrt{m_0/2t}$$

When $\phi_3(H) = 0.746\ H < \phi_2(H)$

$$v_3 = 0.384 \sqrt{m_0 / t}$$

Thus

$$v_1/v_2 = 1.084, \qquad v_2/v_3 = 1.031$$

It is evident that when $\phi_1(H) \geqslant \phi(H) \geqslant \phi_2(H)$ the error resulting from the substitution of $\phi(H)$ by functions $\phi_1(H)$ or $\phi_2(H)$ will not exceed 8.4%, while when $\phi_2(H) \geqslant \phi(H) \geqslant \phi_3(H)$ it will not exceed 3.1%.

In conclusion, let us note that, since the unsteady filtration of gas is described by an equation of the form (1.1), the method we have proposed may be used in a number of cases for the evaluation of approximate solutions of problems of underground gas dynamics.

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Translated by C.E.T.